MTH 201
Multivariable calculus and differential equations
Practice problems

Vectors, dot product, and cross product

1. Find the component form and length of vector $\overrightarrow{PQ}$ with the following initial point and terminal point

   (a) $P(-4,1,-3)$ and $Q(-2,0,2)$

   (b) $P(-1,3,1)$ and $Q(4,7,0)$

   (c) $P(1,1,0)$ and $Q(1,1,1)$

   Also find the vector $\overrightarrow{OR}$, where $\mathbf{0}$ is the origin and R is the midpoint of $\overrightarrow{PQ}$.

2. Find vectors $\mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}$, and $\mathbf{2u}+\mathbf{3v}$ for the following set of vectors

   (a) $\mathbf{u} = \langle 1,0,1\rangle,\ \mathbf{v} = \langle 1,1,1\rangle$

   (b) $\mathbf{u} = \langle 3,-5,7\rangle,\ \mathbf{v} = \langle 5,0,9\rangle$

   (c) $\mathbf{u} = \langle 0,-1,1\rangle,\ \mathbf{v} = \langle 1,-1,0\rangle$

3. Find the unit vector which points in the direction of

   (a) $\mathbf{u} = \langle 3,-5,7\rangle$

   (b) $\mathbf{u} = \langle 5,0,9\rangle$

   (c) $\mathbf{u} = \langle 3,4,5\rangle$

4. Find the angle between vectors

   (a) $\mathbf{u} = 2\mathbf{i}+\mathbf{j}$ and $\mathbf{v} = \mathbf{i}+2\mathbf{j}-\mathbf{k}$

   (b) $\mathbf{u} = 4\mathbf{i}-\mathbf{j}+4\mathbf{k}$ and $\mathbf{v} = -3\mathbf{i}-3\mathbf{j}-4\mathbf{k}$

   (c) $\mathbf{u} = \mathbf{i}+\sqrt{2}\mathbf{j}-\sqrt{2}\mathbf{k}$ and $\mathbf{v} = -\mathbf{i}+\mathbf{j}+\mathbf{k}$

5. Find the measures of angles of the triangle ABC whose vertices are

   (a) $A(-1,0), B(2,1)$, and $C(1,-2)$

   (b) $A(0,0), B(3,5)$, and $C(5,2)$

   (c) $A(0,1), B(1,0)$, and $C(1,1)$.

6. Find vector projection of $\mathbf{u}$ onto $\mathbf{v}$, where

   (a) $\mathbf{u} = 5\mathbf{i}+2\mathbf{j}$ and $\mathbf{v} = \mathbf{i}-3\mathbf{j}$

   (b) $\mathbf{u} = \mathbf{i}+\mathbf{j}+\mathbf{k}$ and $\mathbf{v} = \mathbf{i}$

   (c) $\mathbf{u} = \mathbf{i}+\mathbf{j}$ and $\mathbf{v} = \mathbf{i}-2\mathbf{j}$

7. Find unit vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ in the plane such that $\mathbf{u} = \mathbf{v}+\mathbf{w}$.

8. Verify Cauchy Schwartz inequality and triangle inequality for vectors $\mathbf{u}$ and $\mathbf{v}$ defined in Question 6.

9. Find the cross product of vectors $\mathbf{u}$ and $\mathbf{v}$, where
   (a) $\mathbf{u} = \mathbf{i}$ and $\mathbf{v} = \mathbf{j}$
   (b) $\mathbf{u} = \mathbf{j}$ and $\mathbf{v} = \mathbf{k}$
   (c) $\mathbf{u} = \mathbf{k}$ and $\mathbf{v} = \mathbf{i}$
   (d) $\mathbf{u} = 5\mathbf{i} + 2\mathbf{j}$ and $\mathbf{v} = \mathbf{i} - 3\mathbf{j}$
   (e) $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = \mathbf{i}$
   (f) $\mathbf{u} = \mathbf{i} + \mathbf{j}$ and $\mathbf{v} = \mathbf{i} - 2\mathbf{j}$

10. Let $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ be vectors in the space. Then show that
   (a) $\mathbf{u} \times \mathbf{v} = (\mathbf{u} \cdot (\mathbf{v} \times \mathbf{i}))\mathbf{i} + (\mathbf{u} \cdot (\mathbf{v} \times \mathbf{j}))\mathbf{j} + (\mathbf{u} \cdot (\mathbf{v} \times \mathbf{k}))\mathbf{k}$
   (b) $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = 0$
   (c) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
   (d) $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$
   (e) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = \mathbf{0}$
   (f) $|\mathbf{u} \times \mathbf{v}|^2 = |\mathbf{u}|^2|\mathbf{v}|^2 - |\mathbf{u} \cdot \mathbf{v}|^2$

11. Use vectors to prove that

$$(a^2 + b^2)(c^2 + d^2) \geq (ac + bd)^2$$

   for any four real numbers a, b, c, and d.

12. Let $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ be non-zero vectors in the space. Use dot product and/or cross product to find
   (a) a vector orthogonal to $\mathbf{u} \times \mathbf{v}$ and $\mathbf{u} \times \mathbf{w}$
   (b) a vector orthogonal to $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} + \mathbf{w}$
   (c) a vector of length $|\mathbf{u}|$ in the direction of $\mathbf{v}$
   (d) the area of parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$
   (e) the volume of parallelopiped determined by $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$.

13. Using vectors find the area of triangle determined by points
   (a) $P(1, -1, 2), Q(2, 1, 3), R(-1, 2, -1)$
   (b) $P(1, 1, 1), Q(2, 1, 3), R(3, -1, 1)$
   (c) $P(1, 1, 1), Q(0, 0, 1), R(0, 1, 0)$

Equation of line and plane

14. Find the parametric and vector equation of line passing through $P(1,1,1)$ and parallel to
    (a) $X-$axis
    (b) $Y-$axis
    (c) $Z-$axis.

15. Find the equation of plane
    (a) passing through points $(1,1,-1),(2,0,2)$, and $0,-2,1)$
    (b) passing through point $(2,4,5)$ and perpendicular to the line
    $$x=5+t,\ y=1+3t,\ z=4t,\ t\in\mathbb{R}$$
    (c) passing through point $(1,4,0)$ and perpendicular to $Z-$axis.
    (d) determined by the intersecting lines
    $$\begin{aligned} L1 &: x=-1+t,\ y=2+t,\ z=1-t,\ t\in\mathbb{R} \\ L1 &: x=1-4s,\ y=1+2s,\ z=2-2s,\ s\in\mathbb{R} \end{aligned}$$
    (e) passing through $(2,1,-1)$ and perpendicular to the line of intersection of the planes
    $$2x+y-z=3 \text{ and } x+2y+z=2.$$

16. Find the angle between the planes
    (a) $x+y=1$ and $2x+y-2z=2$.
    (b) $3x-2y-6z=10$ and $4x-y+8z=12$.

17. Find the equation of line of intersection of planes
    (a) $x+y+z=1$ and $x+y=2$.
    (b) $3x-6y-2z=3$ and $2x+y2z=2$.

18. Find the distance between
    (a) the point $(2,1,-1)$ and the plane $x+2y+2z=-5$
    (b) the point $(1,1,-5)$ and the plane $12x+136y+5z=-2$
    (c) the point (a,b,c) and the plane passing through $(1,0,0),(0,1,0)$, and $(0,0,1)$.

19. Show that two planes $ax+by+cz+d_1=0$ and $ax+by+cz+d_2=0$ are parallel, and that the distance between two such planes is
    $$\frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}.$$

Vector valued functions

20. Sketch the following curves
    (a) $\overrightarrow{r(t)} = \langle \cos t, \sin t \rangle,\ 0 \le t \le 2\pi$
    (b) $\overrightarrow{r(t)} = \langle 2\cos t, 3\sin t \rangle,\ 0 \le t \le 3\pi$
    (c) $\overrightarrow{r(t)} = \langle \cos t, \sin t, 2t \rangle,\ 0 \le t \le 2\pi$
    (d) $\overrightarrow{r(t)} = \langle t, 2t, 3t \rangle,\ 0 \le t \le 2$

21. Find the points of intersection of
    (a) the curve $\overrightarrow{r(t)} = \langle t, 2t, 5t^2 \rangle,\ 0 \le t \le 2\pi$, and the paraboloid $z = x^2 + y^2$
    (b) the curve $\overrightarrow{r(t)} = \langle \cos t, \sin t, t \rangle,\ 0 \le t \le 2\pi$, and the sphere $x^2 + y^2 + z^2 = 5$.

22. Find the vector valued function that represents the curve of intersection of $\frac{x^2}{4} + \frac{y^2}{9} = 1$ and
    (a) $z = 1$ (b) $y + z = 2$ and (c) $z = x^2$.

23. Show that
    (a) $\lim_{t \to t_0} \left(\overrightarrow{r_1(t)} \cdot \overrightarrow{r_2(t)} \right) = \lim_{t \to t_0} \overrightarrow{r_1(t)} \cdot \lim_{t \to t_0} \overrightarrow{r_2(t)}$
    (b) $\lim_{t \to t_0} \left(\overrightarrow{r_1(t)} \times \overrightarrow{r_2(t)} \right) = \lim_{t \to t_0} \overrightarrow{r_1(t)} \times \lim_{t \to t_0} \overrightarrow{r_2(t)}$

24. Show that the function $\overrightarrow{r(t)} = \langle \cos t, \sin t, [t] \rangle$ is not continuous at integer points, where $[t]$ denotes the greatest integer part of t.

25. Let f, g, h be real functions defined over $\mathbb{R}$. Show that $\overrightarrow{r(t)} = \langle f(t), g(t), h(t) \rangle$ is continuous at $t = t_0$ if and only if f, g, and h are continuous at $t = t_0$.

26. Find $\overrightarrow{r'(t)}, \overrightarrow{r''(t)}$, and $\overrightarrow{r'(t)} \times \overrightarrow{r''(t)}$ for the following functions
    (a) $\overrightarrow{r(t)} = \langle \cos t, \sin t, 2t \rangle$
    (b) $\overrightarrow{r(t)} = \langle \cos^3 t, \sin^3 t, e^t \rangle$
    (c) $\overrightarrow{r(t)} = \langle t, t^2, t^3 \rangle$

27. If $\mathbf{u} = \overrightarrow{r(t)} \cdot \left(\overrightarrow{r'(t)} \times \overrightarrow{r''(t)} \right)$, then show that $\mathbf{u}' = \overrightarrow{r(t)} \cdot \left(\overrightarrow{r''(t)} \times \overrightarrow{r'''(t)} \right)$.

28. Show that

$$\frac{d}{dt} \left(\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \right) = \mathbf{u}' \cdot (\mathbf{v} \times \mathbf{w}) + \mathbf{u} \cdot (\mathbf{v}' \times \mathbf{w}) + \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}').$$

29. If $\overrightarrow{r(t)} \neq 0$, show that

$$\frac{d}{dt}|\overrightarrow{r(t)}| = \frac{\overrightarrow{r(t)} \cdot \overrightarrow{r'(t)}}{|\overrightarrow{r(t)}|}.$$

30. Let $\overrightarrow{r(t)}$ be a differentiable vector valued function of constant length, i.e., $|\overrightarrow{r(t)}| = constant\ \forall t$. Then prove that $\overrightarrow{r(t)}$ is orthogonal to $\overrightarrow{r'(t)}$.

31. Let $\overrightarrow{r(t)}$ be a differentiable vector valued function such that $\overrightarrow{r(t)} \cdot \overrightarrow{r(t)} = 0\ \forall t$. Then show that $\overrightarrow{r(t)}$ is of constant length.

Curves, arc length, curvature

32. Find arc length of the curve
    (a) $\overrightarrow{r(t)} = \langle 2t, 3t^2, 4t^3 \rangle,\ 0 \leq t \leq 1$
    (b) $\overrightarrow{r(t)} = \langle t, e^{-t}, te^{-t} \rangle,\ 1 \leq t \leq 3$
    (c) $\overrightarrow{r(t)} = \langle \sin t, \cot t, \tan t \rangle,\ 0 \leq t \leq \pi/4$.

33. Reparametrize the curve with respect to arc length measured from the point where $t = 0$ in the direction of increasing t
    (a) $\overrightarrow{r(t)} = \langle 2t, 1 - 3t, 4 - 5t \rangle$
    (b) $\overrightarrow{r(t)} = \langle e^{2t} \cos 2t, 2, e^{2t} \sin 2t \rangle$
    (c) $\overrightarrow{r(t)} = \langle 3 \sin t, 4 \cot t, 5t \rangle$.

34. Suppose you start moving from the point $(0, 0, 3)$ and move 5 units along the curve $\overrightarrow{r(t)} = \langle 3 \sin t, 4t, 3 \cos t \rangle$ in the positive direction. Where are you now?

35. Reparametrize the curve $\overrightarrow{r(t)} = \langle \frac{2}{t^2+1} - 1, \frac{2t}{t^2+1} \rangle$ with respect to arc length measured from the point $(1, 0)$ in the direction of increasing t. Express the reparametrization in its simplest form.

36. Find unit tangent vector $\overrightarrow{T}$, unit normal vector $\overrightarrow{N}$, and binormal vector $\overrightarrow{B}$ for curves defined in Questions 32 and 33.

37. Find the curvature of $\overrightarrow{r(t)}$ at the specified point
    (a) $\overrightarrow{r(t)} = \langle t^2, \log t, t \log t \rangle$ at $(1, 0, 0)$
    (b) $\overrightarrow{r(t)} = \langle t, t^2, t^3 \rangle$ at $(1, 1, 1)$
    (c) $\overrightarrow{r(t)} = \langle \sin t, \cos t, \sin 5t \rangle$ at $(1, 0, 0)$.

38. Use vectors to find the curvature of the following curves at a general point
(a) $y = x^2$ (b) $y = \tan x$ (c) $y = xe^x$.

39. Find an equation of a parabola that has curvature 4 at $(0, 0)$.

40. Show that the circular helix $\overrightarrow{r(t)} = \langle a \sin t, a \cos t, bt \rangle$, where $a, b > 0$, has constant curvature and constant torsion.

Function of two and three variables
Limit and continuity

41. Describe level curves of the following functions of two variables
   (a) $f(x, y) = x + y + 1$
   (b) $f(x, y) = 4x^2 + y^2 + 1$
   (c) $f(x, y) = x^2$.

42. Describe level surfaces of the following functions of three variables
   (a) $f(x, y, z) = x^2 + y^2 + z^2$
   (b) $f(x, y, z) = x + 2y + 3z$
   (c) $f(x, y, z) = x^2 + z^2$.

43. Compute $\lim_{x\to 0}\left(\lim_{y\to 0} f(x, y)\right)$, $\lim_{y\to 0}\left(\lim_{x\to 0} f(x, y)\right)$, and $\lim_{(x,y)\to(0,0)} f(x, y)$, if they exist
   (a) $f(x, y) = x^2 + 5x + 2y + 9$
   (b) $f(x, y) = \sin xe^y$
   (c) $f(x, y) = \frac{x^3+2x^2+xy^2+2y^2}{x^2+y^2}$
   (d) $f(x, y) = \frac{e^{2y}}{x+1}$
   (e) $f(x, y) = \frac{x^2y+y^3}{x^2+2y^2}$
   (f) $f(x, y) = \frac{x^2y^2}{(|x|+|y|)^3}$
   (g) $f(x, y) = \frac{x^2-xy}{\sqrt{x}-\sqrt{y}}$

44. Prove that there exists a $\delta > 0$ such that whenever $x^2 + y^2 \le \delta^2$, we have
$$|x^2 + y^2 + 3xy| < \frac{1}{1000}.$$

45. Show that $f(x, y) = \frac{x+y}{x-y}$ is continuous at $(1, 2)$.

46. Suppose $X, Y \in \mathbb{R}^3$ be such that $X \neq Y$. Show that there exists a continuous function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $f(X) = 0$, $f(Y) = 1$, and $0 \le f(Z) \le 1$ for all $Z \in \mathbb{R}^3$.

Partial derivatives, directional derivatives, and differentiation

47. Compute $f_x, f_y, f_z, f_{xy}, f_{yz}$, and f_{zx} for each of the following functions
    (a) $f(x,y,z) = x^2y + y^3z$
    (b) $f(x,y,z) = \cos(xy) + y^3\sin(xz^2)$
    (c) $f(x,y,z) = \cos(x^2y) - \cos(x^2 - yz)$
    (d) $f(x,y,z) = \sqrt{1 - x^2 - y^2 - z^2}$.

48. Find an equation of tangent plane to the given graph at the specified point
    (a) $z = \cos x \cos y$; at $(0, \pi/2, 0)$
    (b) $z = x^2 + 4y^2$; at $(2, -1, 8)$
    (c) $z = \frac{1}{xy}$; at $(1, 1, 1)$.

49. Find the linear approximation of given function at the specified point
    (a) $f(x,y) = e^x + \sin xy$; at $(1, 3)$
    (b) $z = x^2 + 4y^2$; at $(2, 1)$.

50. Verify the chain rule for
    (a) $z = xy;\ x = t^2, y = t^3$
    (b) $z = xy;\ x = t^2 - s^2, y = t^2 + s^2$
    (c) $w = xe^{yz};\ x = e^t, y = t, z = \sin t$
    (d) $w = f(x,y,z);\ x = r\cos\theta, y = r\sin\theta, z = z$
    (e) $w = f(x,y,z);\ x = r\cos\theta\sin\phi, y = r\sin\theta\sin\phi, z = r\cos\phi$.

51. Compute the directional derivative $D_{\mathbf{u}}f(x_0, y_0)$ where
    (a) $f(x,y) = xe^{xy} + y,\ \mathbf{u} = (\cos 2\pi/3, \sin 2\pi/3), (x_0, y_0) = (2, 0)$
    (b) $f(x,y) = x\cos y,\ \mathbf{u} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (x_0, y_0) = (0, 0)$

52. Let $f(x,y) = \frac{\cos x + e^{xy}}{x^2 + y^2}$. Show that f is differentiable at all points $(x,y) \neq (0,0)$.

53. Let $f(x,y) = \frac{y}{|y|}\sqrt{(x^2 + y^2)}$, if $y \neq 0$ and $f(x,y) = 0$ if $y = 0$. Show that f is continuous at $(0,0)$, it has directional derivatives in every direction at $(0,0)$, but it is not differentiable at $(0,0)$.

54. Let $f(x,y) = \frac{1}{2}(||x| - |y|| - |x| - |y|)$. Is f continuous at $(0,0)$? Which directional derivatives of f exist at $(0,0)$? Is f differentiable at $(0,0)$?